MIXED STRATEGIES

We had seen games for which a PSNE may not exist.
Moreover, our games have deterministic strategies, which is not always the case.

MIXED STRATEGIES

Each player has a probability distribution over available actions.

Notation

Given a set A, $\Delta A = \{ p \in [0,1]^{|A|} \mid \sum p_a = 1 \}$

set of all probability distributions over elements of A

- σ_i is a mixed Strategy for player i IF $\sigma_i \in \Delta(S_i)$
  - Note that players choose strategies independently as we deal with non-cooperative games

$\Rightarrow$ Joint prob. for 1 using s_1 and 2 using s_2 = $\sigma_1(s_1)\sigma_2(s_2)$

$\Rightarrow$ Utility of 1 with strategy profile (σ_i, σ_{-i})

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \sigma_1(s_1) \ldots \sigma_n(s_n) u_i(s_1, \ldots, s_n)$$

Simply add up all cases' utilities
"Expected Utility"

$\Rightarrow$ Similarly, $u_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i, s_{-i})$

MIXED STRATEGY NASH EQUILIBRIUM

Defn. A profile $(\sigma_i^*, \sigma_{-i}^*)$ is MSNE iff

$$u_i(\sigma_i', \sigma_{-i}^*) \leq u_i(\sigma_i^*, \sigma_{-i}^*) \quad \forall i \in N, \forall \sigma_i' \in \Delta(S_i)$$

Theorem A mixed strategy profile is MSNE IFF

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \quad \forall i \in N, \forall s_i \in S_i$$

Powerful, as we only need to check finite cases to prove (σ_i, σ_{-i}) is MSNE

Proof Forward direction is trivial, and follows from definition.

Backward

Take arbitrary $\sigma_i \Rightarrow u_i(\sigma_i, \sigma_{-i}^*) = \sum_{s_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i}^*) \leq \sum_{s_i} \sigma_i(s_i) u_i(\sigma_i^*, \sigma_{-i}^*)$

$$\leq u_i(\sigma_i^*, \sigma_{-i}^*) \sum_{s_i} \sigma_i(s_i)$$

$$\leq u_i(\sigma_i^*, \sigma_{-i}^*)$$

Support of a mixed strategy

For a mixed strategy σ_i, the subset of strategy space of i where σ_i has positive mass.

$$\delta(\sigma_i) = \{s_i \in S_i \mid \sigma(s_i) > 0\}$$

Characterization Theorem

A strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is MSNE $\boxed{\text{IFF}}$ $\forall i \in N$

1) $u_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$
2) $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)$ $\forall s_i \in \delta(\sigma_i^*)$, $s_i' \notin \delta(\sigma_i^*)$

This theorem is used to find MSNE

Proof

Observe that $\max\limits_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}) = \max\limits_{s_i \in S_i} u_i(s_i, \sigma_{-i})$

Expand

$\sum \sigma_i(s_i) u_i(s_i, \sigma_{-i})$

Just pick the strategy with highest utility!

Furthermore, $\max\limits_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) = \underbrace{\max\limits_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)}_{\text{follows from prev}} = \underbrace{\max\limits_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*)}_{\text{Proof by Contradiction}}$

<u>Forward Direction</u>, given $(\sigma_i^*, \sigma_{-i}^*)$ is MSNE

$\Rightarrow u_i(\sigma_i^*, \sigma_{-i}^*) = \max$

Algorithmic Way to find MSNE

Given NFG $G = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$,

total combinations of support $K = (2^{|S_1|}-1)\dots(2^{|S_n|}-1)$

For every support profile, solve the following feasibility problem :-
$(X_1 \times X_2 \times \dots)$

1) $w_i = \sum_{s_{-i}\in S_{-i}} \underbrace{\left[\prod_{j\neq i} \sigma_j(s_j)\right]}_{\sigma_{-i}(s_{-i})} u_i(s_i, s_{-i})$; $\forall s_i \in X_i, \forall i \in N$

2) $w_i \geq \sum_{s_{-i}\in S_{-i}} \left[\prod_{j\neq i} \sigma_j(s_j)\right] u_i(s_i, s_{-i})$; $\forall s_i \in S_i \setminus X_i, \forall i \in N$

$\sigma_j(s_j) \neq 0, \ \sum \sigma_j(s_j) = 1$

This is not a linear program unless $n=2$.

MSNE AND DOMINANCE

Theorem: If a pure strategy s_i is dominated by a mixed strategy σ_i Then s_i is picked with probability zero. $\Rightarrow$ Remove s_i WLOG

Existance theorem – Every finite game has MSNE