CS6001 - Game Theory

Part 1

Introduction,

NFG + Nash Eqb.

Introduction To Game Theory

- We shall focus on Algorithmic design and analysis and use game theory to get us there.

  That is, we shall design a game such that a reasonable outcome is achieved.

- We shall be representing games in the following manner:-

A\B	B_1	B_2
A_1	5,5	0,6
A_2	6,0	1,1

→ The Prisoner's Dillemma

  The first element of the tuple is the utility of player A, and second element is the utility for Player B.

- A game is a strategic interaction between players with a strategy.
  mapping from state → action

Normal Form Games - games where each player makes a choice and the game ends once every player has made a choice

- Agents are assumed to be:-
  - Rational - desires the highest utility
  - Intelligent - knows rules of the game and picks actions assuming other rational and intelligent people.
    - has enough info to compute the "optimal" solution

  has common knowledge

* A Game Of Chess

- The natural question that we pose are:-
  1) Does W/B have a winning strategy?
  2) Does either have a strategy to ensure a draw?
  3) Are neither possible?

- A winning strategy s_W^* is such that $\forall\, s_B$, (s_W^*, s_B) always ends in W winning
  Draw guranteeing strategy " , (s_W^*, s_B) is always either a draw or A wins

Theorem

Either W has a Winning Strategy
OR B has "
OR W/B have a Draw-Guaranteeing Strategy

Proof Can prove quite easily using induction over subtrees and node-count.

Construct a game tree with a node being a state

$\Gamma(x)$ - subtree rooted at x, including x

n_x - number of nodes in $\Gamma(x)$ $\Rightarrow n_x = 1$ if x is terminal

Induction over n_x

Basis of Induction

$n_x = 1 \Rightarrow$ consider a terminal game state.

$\Rightarrow$ The statement is vacuously true

Inductive Hypothesis

Consider that the statement holds for all nodes y with $n_y < K$

Inductive Step

Work from the bottom-up.

Consider a node x with $n_x > 1$, and y to be a descendant of x.

$n_y < n_x$

WLOG, assume it is white's turn at x, and black's turn at y.

Case 1) $\forall y$, Black has a winning strategy

$\Rightarrow$ Nothing at x can stop Black from winning

$\Rightarrow$ Black has "won" at x. - ②

Case 2) $\exists y$, White has a winning strategy

$\Rightarrow$ Simply pick that!

$\Rightarrow$ White has a winning strat at x. — ①

Case 3) $\exists y$, Black has no winning strat and

$\forall y$, White has no winning strat

$\Rightarrow$ From hypothesis, both B/W have a strat to draw

$\Rightarrow$ Pick that!

Now both B/W can reach a draw from x — ③

All cases covered.

Statement proven using strong mathematical induction.

* Representing Normal form games

$N = \{1, 2, \dots n\}$ - set of players

S_i - set of strategies for i'th player

S_{-i} - " everyone except i'th player

$(s_i, s_{-i}) \leftarrow S = \underset{i \in N}{\times} S_i$ - Set of strategy profiles

finite strategy, if S_i is finite $\forall i \in N$

$u_i : S \to \mathbb{R}$ - Utility for player i

NFG - $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

Domination In NFGs

Loosely speaking, a strategy is said to be dominated when there exists some other strategy which is very clearly better than the current one.

- Formally, $s_i' \in S_i$ is said to be strictly dominated if $\exists s_i \in S_i$ such that

$$\forall s_{-i} \in S_{-i}, \quad u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

- Similarly, $s_i' \in S_i$ is said to be weakly dominated if $\exists s_i \in S_i$ such that

$$\forall s_{-i} \in S_{-i}, \quad u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

$$\text{and } \exists s_{-i} \in S_{-i}, \quad u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

not really "dominating" if both have same utilities for all s_{-i}

- Similarly, a strategy s_i is STRICTLY/WEAKLY DOMINANT if it strictly/weakly dominates all other strategies $s_i' \in S_i \setminus \{s_i\}$

Dominant Strategy Equilibrium

A profile S^* is the strictly/weakly dominant strategy eqb. if every $s_i^* \in S$ is strictly/weakly dominant.

Rational Outcomes of a game

A player would never play a dominated strategy.

⇒ Can we simply eliminate these strategies to get the rational outcome?

NO.

- Order of elimination for SDS does not matter
- However, order does matter for WDS! A possibility of eliminating rational outcomes exists!

• Moreover, dominant strategies / DSE is not guaranteed to exist!

P1 \ P2	L	R
L	1,1	0,0
R	0,0	1,1

Co-ordination game

To deal with these drawbacks, we introduce a new eqb.

Nash Equilibrium

- A strategy profile (s_i^*, s_{-i}^*) is a Pure Strategy Nash Equilibrium (PSNE) if

$$\forall i \in N, \forall s_i \in S_i \; ; \quad u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

analogous to a local maxima!

Note that (L,L) and (R,R) are PSNE in the co-ordination game.

* Best Response View

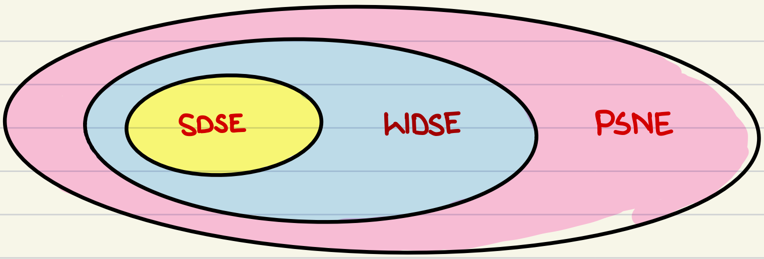
The best response of player i against s_{-i} is a strategy that gives the max utility.

$$B(s_{-i}) = \left\{ s_i \in S_i \;\middle|\; u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \; \forall s_i' \in S_i \right\}$$

It can be proven that PSNE is a profile (s_i^*, s_{-i}^*) such that

$$\forall i \in N, \quad s_i^* \in B(s_{-i}^*)$$

No rational player would deviate from it!



Max-Min Strategies

- We had assumed all players to be rational when computing PSNE. This assumption is risky as we might get a terrible result if the other player uses an un-optimal strategy.

1\2	L	R
T	3,1	2,0
M	-10,0	5,1

→ (M,R) is PSNE, but risky for P1
if P2 plays L, P1 gets -10 which is quite bad

- The worst-case optimal choice is called the max-min optimal strategy.

for a given s_i, what is the worst case utility?

$$s_i \in \underset{s_i \in S_i}{\operatorname{argmax}} \; \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

→ maximize this worst case utility

- The max-min value is the least possible utility following the max-min strategy.

$$\underline{v_i} = \max_{s_i \in S_i} \; \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

$$\Rightarrow u_i(s_i^{\text{maxmin}}, s_{-i}) \geq \underline{v_i} \quad \forall s_{-i} \in S_{-i}$$

meaning looking at dominant strategies is "risk free"

Theorem If s_i^* is a dominant strategy for player i, then it is also a max-min strategy

Proof

Case 1 - Strictly dominant

From the defn itself, $u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i})$ $\forall s_i \in S_i \setminus \{s_i^*\}$ $\forall s_{-i} \in S_{-i}$

Define a new function s_{-i}^{min} which takes s_i as argument and gives the worst possible s_{-i}

$$s_{-i}^{min}(s_i) = \underset{s_{-i} \in S_{-i}}{\text{argmin}} \; u_i(s_i, s_{-i})$$

From the defn, it holds that

$$u_i(s_i^*, s_{-i}^{min}(s_i')) > u_i(s_i', s_{-i}^{min}(s_i')) \quad \forall s_i' \in S \setminus \{s_i^*\}$$

That is, for every $s_i' \neq s_i^*$

s_i^* gets the best possible utility even in the worst case scenario

$\Rightarrow$ s_i^* is max-min strategy

Similar proof for a weakly dominant strategy works.

Theorem Every PSNE $s^* = (s_1^*, \ldots, s_n^*)$ of NFG satisfies $u_i(s^*) \geq \underline{v}_i$

that is, utility in every PSNE would be atleast the worst-case max-min strategy's utility

Proof

From def. of PSNE,

$$u_i(s_i^*, s_{-i}^*) = \max_{s_i} u_i(s_i, s_{-i}^*) \geq \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$ from def. of min

$$\geq \underline{v}_i$$

* ITERATED ELIMINATION OF DOMINATED STRATEGIES

We shall now look at effects of iterated elimination on :-

① PSNE

② $\underline{v}_i$ - max min value

② **Theorem**

Consider an NFG = $\langle N, S_i, u_i \rangle$ and let $\hat{s}_j \in S_j$ be a dominated strategy.
Let the residual game after removing $\hat{s}_j$ be $\hat{G}$.

The maxmin value for j in both G and $\hat{G}$ are equal!

- the maxmin value might change for other players!

Proof

Let maxmin in G for j = $\underline{v}_j = \max\limits_{s_j} \min\limits_{s_{-j}} u_j(s_j, s_{-j})$; $s_j \in S_j$

" $\hat{G}$ for j = $\hat{\underline{v}}_j = \max\limits_{s_j} \min\limits_{s_{-j}} u_j(s_j, s_{-j})$; $s_j \in S_j \setminus \{\hat{s}_j\}$

Proof by Contradiction

Assume $\underline{v}_j \neq \hat{\underline{v}}_j \Rightarrow \hat{\underline{v}}_j < \underline{v}_j \Rightarrow \hat{s}_j$ was the only max-min strategy

$\Rightarrow \hat{s}_j = \arg\max\limits_{s_j} \min\limits_{s_{-j}} u_j(s_j, s_{-j})$

however $\hat{s}_j$ is dominated $\Rightarrow$ argmax CANNOT yield $\hat{s}_j$

CONTRADICTION!

① **Theorem**

Consider a NFG G and let $\hat{G}$ be the game after elimination of a not necessarily dominated strategy. If profile s^* is PSNE in G, and it survives in $\hat{G}$,
Then s^* is PSNE in $\hat{G}$ as well!

Proof

s^* is PSNE in G $\Rightarrow$ $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \quad \forall i \in N, \forall s_i' \in S_i$

Let player j's s_j be removed

$j \neq i$ as s^* exists in $\hat{G}$

$\Rightarrow$ inequality of maxima unaffected!

Theorem

Let $\hat{s}_j$ be a weakly dominated strategy for G, and eliminating it yields $\hat{G}$.

Every PSNE of $\hat{G}$ is also a PSNE for G — New PSNE cannot form! Old ones may be removed.

Proof

s_j is weakly dominated

$\Rightarrow \exists\, s_j' \in S_j \setminus \{s_j\}$ such that $u_j(s_j', s_{-j}) \geq u_j(s_j, s_{-j}) \quad \forall s_{-j} \in S_{-j}$

Let the profile s^* be a PSNE in $\hat{G}$

$\Rightarrow u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \quad \forall i \neq j, \forall s_i \in S_i$

$i = j \Rightarrow s_j' \in S_j \setminus \{s_j\}$

For this to be a PSNE in G, we need — Rest all cases are covered here

$u_j(s_j^*, s_{-j}^*) \geq u_j(s_j, s_{-j}^*)$

Using the fact that s_j is dominated;

$$\exists s_j' \in S_j \text{ st. } u_j(s_j^*, s_{-j}^*) \underbrace{\ge}_{\text{PSNE}} u_j(s_j', s_{-j}^*) \underbrace{\ge}_{\text{WD}} u_j(s_j, s_{-j}^*)$$

$\therefore s^*$ is PSNE in G as well.

* SUMMARY

- Removing SDS has no effect on PSNE
- " WDS may remove PSNE, but never adds a new PSNE
- Maxmin of a player is unaffected by removing either

MATRIX GAMES (TWO PLAYER ZERO SUM)

- A special class of NFGs with
  - $N = 2$
  - $u_1 + u_2 = 0 \;\; \forall s \in S$

	L	R
L	2,-2	-3,3
R	0,0	1,-1

	L	R	
L	2	-3	-3
R	0	1	0 → 0
	2	1 → 1	

Grid elements are wrt player 1

The calculation of max-min for player 1 is unchanged.

Player 2 needs to compute min-max instead!

- SADDLE POINT for the matrix is when a value is max for P1 (along row) but min for P2 (along col.)

In a matrix game with utility u,

(s_1, s_2) is a saddle point IFF (s_1, s_2) is a PSNE

- Similar to NFG, define :-

max-min $\underline{v} = \max_{s_1} \min_{s_2} u(s_1, s_2)$

min-max $\overline{v} = \min_{s_2} \max_{s_1} u(s_1, s_2)$

$\rightarrow$ $\overline{v} \ge \underline{v}$ ALWAYS!

Theorem A matrix game has PSNE $\boxed{\text{iff}}$ $\overline{v} = \underline{v} = u(s_1, s_2)$; and (s_1, s_2) is also a PSNE