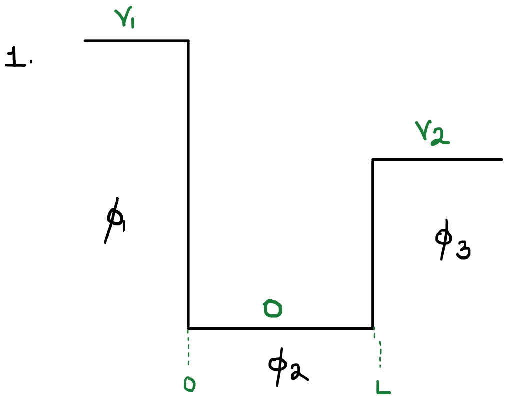
Tut Sheet 8

1.



Bound states $\Rightarrow E < \min(V_1, V_2)$

$$\phi_1 = Ae^{-\alpha_1 x} + Be^{\alpha_1 x}$$

$$\phi_2 = Ce^{-ikx} + De^{ikx}$$

$$\phi_3 = Ee^{-\alpha_2 x} + Fe^{\alpha_2 x}$$

$$\alpha_1^2 = \frac{2m(V_1 - E)}{\hbar^2}$$

$$\alpha_2^2 = \frac{2m(V_2 - E)}{\hbar^2}$$

$$k^2 = \frac{2mE}{\hbar^2}$$

No incidence $\Rightarrow F = 0$ and $A = 0$

Continuity

$(x=0) \Rightarrow \phi_1(0) = \phi_2(0) \Rightarrow B = C + D$ — ①

$(x=L) \Rightarrow \phi_2(L) = \phi_3(L) \Rightarrow Ce^{-ikL} + De^{ikL} = Ee^{-\alpha_2 L}$ — ②

Differentiability

$(x=0) \Rightarrow \alpha_1 B = -ikC + ikD$ — ③

$(x=L) \Rightarrow -ikCe^{-ikL} + ikDe^{ikL} = -\alpha_2 Ee^{-\alpha_2 L}$ — ④

$$\alpha_1 C + \alpha_1 D = -ikC + ikD$$

$$C(\alpha_1 + ik) = D(-\alpha_1 + ik)$$

$$\frac{C}{D} = \frac{-\alpha_1 + ik}{\alpha_1 + ik}$$

$$-\alpha_2 Ce^{-ikL} - \alpha_2 De^{ikL} = -ikCe^{-ikL} + ikDe^{ikL}$$

$$(Ce^{-ikL})(-\alpha_2 + ik) = (De^{ikL})(\alpha_2 + ik)$$

$$\frac{C}{D} = \frac{(e^{2ikL})(\alpha_2 + ik)}{-\alpha_2 + ik}$$

$$\frac{-\alpha_1 + ik}{\alpha_1 + ik} = \frac{(e^{2ikL})(\alpha_2 + ik)}{(-\alpha_2 + ik)} \Rightarrow \alpha_1\alpha_2 - ik(\alpha_1 + \alpha_2) - k^2$$

$$= (e^{2ikL})(\alpha_1\alpha_2 + ik(\alpha_1 + \alpha_2) - k^2)$$

$$\Rightarrow (\alpha_1\alpha_2 - k^2)(e^{2ikL} - 1) + ik(\alpha_1 + \alpha_2)(e^{2ikL} + 1) = 0$$

$$\Rightarrow \quad (\alpha_1\alpha_2 - k^2)\left(\frac{e^{ikL}-e^{-ikL}}{2}\right) + ik(\alpha_1+\alpha_2)\left(\frac{e^{ikL}+e^{-ikL}}{2}\right) = 0$$

$$(\alpha_1\alpha_2 - k^2)(i\sin kL) + (ik)(\alpha_1+\alpha_2)\cos kL = 0$$

$$(\sin kL)(k^2-\alpha_1\alpha_2) = (k)(\alpha_1+\alpha_2)\cos kL \Rightarrow \tan kL = \frac{k(\alpha_1+\alpha_2)}{k^2-\alpha_1\alpha_2}$$ //

When $V_1 \to \infty$, $\alpha_1^2 = \frac{2m(V_1-E)}{\hbar^2} \to \infty$; apply limits to get $\tan kL = \frac{k(1)}{-\alpha_2}$

2. $\phi_1(x) = A\sin(kx) + B\cos(kx)$

By symmetry, $\phi_2(x) = -A\sin(kx) + B\cos(kx)$

at $x = (-2L)$:- $-A\sin(2kL) + B\cos(2kL) = 0$ (why not diff. condition here?)

at $x = (-L)$:- $-\sin(kL) + B\cos(kL) = c$

$Ak\cos kL - Bk\sin(kL) = 0$

$$A\sin(2kL) = B\cos(2kL) \Rightarrow \frac{\sin(2kL)}{\cos(2kL)} = \frac{-\cos(kL)}{\sin(kL)} \Rightarrow \cos(kL) = 0$$

$$A\cos(kL) = -B\sin(kL) \qquad \text{or} \quad \boxed{kL = \frac{2n+1}{2}\pi}$$

(b) $C = -A\sin(kL)$

(c) $Bk\sin(kL) = 0 \to B = 0 \Rightarrow \phi_1(x) = A\sin(kx)$

Normalize:- $\int_{-2L}^{2L} \psi^*\psi \, dx = 1$

$$\Rightarrow 2\int_{-2L}^{-L} A^2\sin^2(kx)\,dx + 2LC^2 = 1$$

$$\Rightarrow A = 1/\sqrt{3L} \quad \text{(taking real)}$$

use to get (b)

(Leaving Rest for exercise)

3. $\psi(x,0) = A(\phi_1 + \phi_2 + \phi_4)$

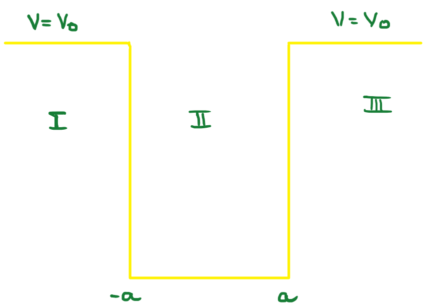
$$\Rightarrow \int \psi\psi^* dx = \int A^2(\phi_1^2 + \phi_2^2 + \phi_4^2)dx = 3A^2 = 1$$

$$\Rightarrow A = \frac{1}{\sqrt{3}} e^{i\theta} = \frac{1}{\sqrt{3}} \text{ (for ease of calculations)}$$

b. $\psi(x,t) = \frac{1}{\sqrt{3}}\left(\phi_1 e^{-\frac{iE_1 t}{\hbar}} + \phi_2 e^{-\frac{iE_2 t}{\hbar}} + \phi_4 e^{-\frac{iE_4 t}{\hbar}}\right)$ Superposition!

c. $\langle E \rangle$ (Solved in slides prev.) $= \dfrac{E_1 + E_2 + E_4}{3}$

4.



$\phi_1 = Ae^{\alpha x}$

$\phi_2 = C\sin kx + D\cos kx$

$\phi_3 = Fe^{-\alpha x}$

$k^2 = \dfrac{2mE}{\hbar^2}$

$\alpha^2 = \dfrac{2m(V_0 - E)}{\hbar^2}$

Symmetric $\Rightarrow$ Even parity & odd parity solutions.

$A = F, C = 0$

$Ae^{-\alpha a} = D\cos(ka)$

$\alpha A e^{-\alpha a} = Dk\sin(ka)$

$\Rightarrow \tan(ka) = \alpha/k$ — ①

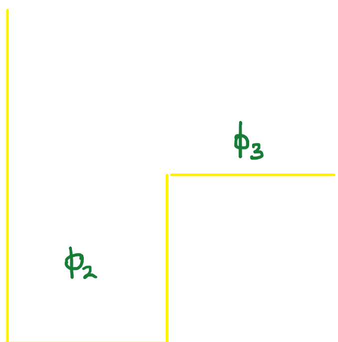
$D = 0, A = -F$

$Ae^{-\alpha a} = -C\sin(ka)$

$A\alpha e^{-\alpha a} = Ck\cos(ka)$

$\tan(ka) = -k/\alpha$ — ②

For Semi-Infinite potential well

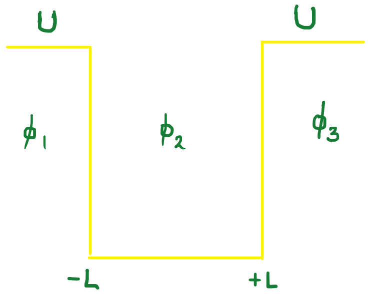


$$\left.\begin{aligned} \phi_2 &= A\cos(kx) + B\sin(kx) \\ \phi_3 &= Ce^{-\alpha x} \end{aligned}\right\}$$ Solve to get $\tan(ka) = -k/\alpha$

$\Downarrow$

ONLY ODD parities Exist now!

5.



$$\phi_1 = Ae^{qx} \qquad \text{where} \qquad q^2 = \frac{2m(U-E)}{\hbar^2}$$

$$\phi_2 = B\cos kx + C\sin kx \qquad k^2 = \frac{2mE}{\hbar^2}$$

$$\phi_3 = Ae^{-qx} \quad \text{(symmetry)}$$

Even parity solutions

$$C = 0$$

at $x = -L$; $Ae^{-qL} = B\cos kL$

$$Aqe^{-qL} = Bk\sin kL$$

$$\Rightarrow \tan kL = q/k \quad \text{—} ①$$

Odd parity Solutions

$$B = 0$$

at $x = (-L)$; $Ae^{-qL} = -C\sin kL$

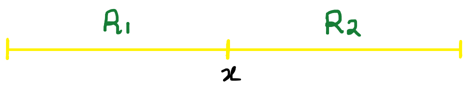
$$Aqe^{-qL} = Ck\cos kL$$

$$\tan kL = -k/q \quad \text{—} ②$$

← Conditions →

b,c are straight-forward now.

6.



If particle could be found in R_2, then $\psi(x) \neq 0$

However, as they are disjoint and particle is in R_1;

$\psi(x) - \psi_1(x) = 0$ at boundary.

CONTRADICTION!

Therefore particle stays in R_1

b. $\psi(x) = \frac{1}{\sqrt{2}}\left[\psi_1(x) + \psi_2(x)\right] \Rightarrow \psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar}\right]$

$$|\psi(x,t)|^2 = \psi(x,t)\psi^*(x,t) = \frac{1}{2}\left[\psi_1 e^{-\frac{iE_1t}{\hbar}} + \psi_2 e^{-\frac{iE_2t}{\hbar}}\right]\left[\psi_1^* e^{\frac{iE_1t}{\hbar}} + \psi_2^* e^{\frac{iE_2t}{\hbar}}\right]$$

$$= \frac{1}{2}\left[|\psi_1|^2 + |\psi_2|^2\right] \qquad R_1 \cap R_2 = \phi$$

(c) If $R_1 \cap R_2 \neq \phi$ then

$$|\psi(x,t)|^2 = \frac{1}{2}\left[|\psi_1|^2 + |\psi_2|^2 + \mathrm{Re}\left[\psi_1(x)\psi_2^*(x)e^{i\frac{E_1-E_2}{\hbar}t}\right]\right]$$